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ScienceDirect
J. Math. Anal. Appl. 324 (2006) 663–668
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On Bernardi's integral operator and the Briot–Bouquet differential subordination ☆

Rosihan M. Ali *, V. Ravichandran, N. Seenivasagan

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

Received 30 August 2005

Available online 18 January 2006

Submitted by S. Ruscheweyh

Abstract

Let $A, B, D, E \in [-1, 1]$. Conditions on A, B, D and E are determined so that

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez} \quad \text{implies } p(z) \prec \frac{1 + Az}{1 + Bz}.$$

The result is applied to Bernardi's integral operator of two classes of analytic functions.

Keywords: Analytic functions; Univalent functions; Starlike functions; Bernardi's integral operator

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $S^*[A, B]$ denote the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leqslant B < A \leqslant 1)$$

☆ The authors gratefully acknowledged support from the research grant IRPA 09-02-05-00020 EAR.
* Corresponding author.
E-mail addresses: rosihan@cs.usm.my (R.M. Ali), vravi@cs.usm.my (V. Ravichandran), vasagan2000@yahoo.co.in (N. Seenivasagan).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.12.033

or the equivalent inequality

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \left|A-B\frac{zf'(z)}{f(z)}\right| \quad (z\in\Delta,\ -1\leqslant B<A\leqslant 1).$$

Functions in $S^*[A,B]$ are called the *Janowski starlike functions* [3,6].

For $0\leqslant\alpha<1$, the class $S^*[1-2\alpha,-1]$ is the familiar class S^*_α of starlike functions of order α, while $S^*[1-\alpha,0]$ is the class $S^*(\alpha)$ of functions $f\in\mathcal{A}$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha \quad (z\in\Delta,\ 0\leqslant\alpha<1).$$

For $0<\alpha\leqslant 1$, $S^*[\alpha,-\alpha]=:S^*[\alpha]$ is the class of functions $f\in\mathcal{A}$ satisfying the condition

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \alpha\left|\frac{zf'(z)}{f(z)}+1\right| \quad (z\in\Delta,\ 0<\alpha\leqslant 1).$$

For this latter class $S^*[\alpha]$, Parvatham proved the following:

Theorem 1.1. [4, Theorem 1, p. 438] *Let $c\geqslant 0$, $0<\alpha\leqslant 1$ and δ be given by*

$$\delta:=\alpha\left[\frac{2+\alpha+c(1-\alpha)}{1+2\alpha+c(1-\alpha)}\right].$$

If $f\in S^*[\delta]$, then the function $F(z)$ given by Bernardi's integral

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{1.1}$$

is in $S^*[\alpha]$.

It is well known [2] that the classes of starlike, convex and close-to-convex functions are closed under Bernardi's integral operator. Since $\delta\geqslant\alpha$, Theorem 1.1 extends the result of Bernardi [2].

Parvatham also considered a similar problem for the class $R[\alpha]$ of functions $f\in\mathcal{A}$ satisfying

$$\left|f'(z)-1\right| < \alpha\left|f'(z)+1\right| \quad (z\in\Delta,\ 0<\alpha\leqslant 1),$$

and proved the following:

Theorem 1.2. [4, Theorem 2, p. 440] *Let $c\geqslant 0$, $0<\alpha\leqslant 1$ and δ be given by*

$$\delta:=\alpha\left[\frac{2-\alpha+c(1-\alpha)}{1+c(1-\alpha)}\right].$$

If $f\in R[\delta]$, then the function $F(z)$ given by Bernardi's integral (1.1) *is in* $R[\alpha]$.

The class $R[\alpha]$ can be extended to the bigger class $R[A,B]$ consisting of all analytic functions $f(z)\in\mathcal{A}$ satisfying

$$f'(z)\prec\frac{1+Az}{1+Bz} \quad (-1\leqslant B<A\leqslant 1),$$

or in other words,

$$\left|f'(z)-1\right| < \left|A-Bf'(z)\right| \quad (z\in\Delta,\ -1\leqslant B<A\leqslant 1).$$

For $0 \leqslant \alpha < 1$, the class $R[1-2\alpha, -1]$ consists of functions $f \in \mathcal{A}$ for which

$$\Re f'(z) > \alpha \quad (z \in \Delta,\ 0 < \alpha \leqslant 1),$$

and $R[1-\alpha, 0] =: R_\alpha$ is the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left|f'(z) - 1\right| < 1 - \alpha \quad (z \in \Delta,\ 0 \leqslant \alpha < 1).$$

When $0 < \alpha \leqslant 1$, the class $R[\alpha, -\alpha]$ is the class $R[\alpha]$ considered by Parvatham [4].

In this paper, we extend Theorems 1.1 and 1.2 to hold true for the more general classes $S^*[A, B]$ and $R[A, B]$, respectively. We shall in fact obtain a more general result relating to the Briot–Bouquet differential subordination, and then apply it to Bernardi's integral operator of the classes $S^*[D, E]$ and $R[D, E]$. The proofs are, however, very computationally involved.

2. A Briot–Bouquet differential subordination

Theorem 2.1. *Let $-1 \leqslant B < A \leqslant 1$ and $-1 \leqslant E \leqslant 0 < D \leqslant 1$. For $\beta \geqslant 0$ and $\beta + \gamma > 0$, let $G := A\beta + B\gamma$, $H := (\beta + \gamma)(D - E)$, $I := (A\beta + B\gamma)(D - E) + (BD - AE)(\beta + \gamma) - kE(A - B)$, $J := (A\beta + B\gamma)(BD - AE)$, and $L := \beta + \gamma + k$. In addition, for all $k \geqslant 1$, let*

$$\left(L^2 + G^2\right)\left[(H + J)I - 4H|J|\right] + 4LGHJ \geqslant LG\left[(H - J)^2 + I^2\right]. \tag{2.1}$$

Further assume that

$$\frac{[\beta(1+A) + \gamma(1+B) + 1](A - B)}{[\beta(1+A) + \gamma(1+B)][D(1+B) - E(1+A)] - E(A-B)} \geqslant 1. \tag{2.2}$$

Let $p(z)$ be analytic in Δ with $p(0) = 1$. If

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez},$$

then

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Proof. Define $P(z)$ by

$$P(z) := p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$$

and $w(z)$ by

$$w(z) := \frac{p(z) - 1}{A - Bp(z)},$$

or equivalently by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.3}$$

Then $w(z)$ is meromorphic in Δ and $w(0) = 0$. We need to show that $|w(z)| < 1$ in Δ. By a computation from (2.3), we get

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{(A - B)zw'(z)}{(1 + Bw(z))[\beta(1 + Aw(z)) + \gamma(1 + Bw(z))]}.$$

Therefore

$$\frac{P(z)-1}{D-EP(z)} = \frac{(A-B)[(\beta+\gamma)w(z)+(A\beta+B\gamma)w^2(z)+zw'(z)]}{[(D-E)+(BD-AE)w(z)][\beta+\gamma+(A\beta+B\gamma)w(z)]-E(A-B)zw'(z)}.$$

Assume that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z|\leqslant|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by [5, Lemma 1.3, p. 28], there exists $k \geqslant 1$ such that $z_0w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this z_0, we have

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| = \left|\frac{(A-B)[L+Gw(z_0)]}{H+Iw(z_0)+Jw(z_0)^2}\right| = (A-B)\big[\varphi(\cos\theta)\big]^{1/2},$$

where

$$\varphi(\cos\theta) := \frac{|L+Ge^{i\theta}|^2}{|He^{-i\theta}+Je^{i\theta}+I|^2} = \frac{L^2+G^2+2LG\cos\theta}{H^2+J^2+I^2+2HJ\cos 2\theta+2I(H+J)\cos\theta}.$$

In view of the fact that

$$\min\{at^2+bt+c\colon -1\leqslant t\leqslant 1\} = \begin{cases} \frac{4ac-b^2}{4a}, & \text{if } a>0 \text{ and } |b|<2a, \\ a-|b|+c, & \text{otherwise,}\end{cases}$$

the function

$$\varphi(t) := \frac{L^2+G^2+2LGt}{4HJt^2+2I(H+J)t+(H-J)^2+I^2}$$

is easily seen to be a decreasing function of $t=\cos\theta$ provided (2.1) holds. Thus we have $\varphi(t) \geqslant \varphi(1) = [(L+G)/(I+J+H)]^2$. Yet another calculation shows that the function $\psi(k) := (L+G)/(I+J+H)$ is an increasing function of k. Since $k \geqslant 1$, we have $\psi(k) \geqslant \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geqslant \frac{[\beta(1+A)+\gamma(1+B)+1](A-B)}{[\beta(1+A)+\gamma(1+B)][D(1+B)-E(1+A)]-E(A-B)},$$

which by (2.2) is greater than or equal to 1. This contradicts that $P(z) \prec (1+Dz)/(1+Ez)$ and completes the proof. □

3. Bernardi's integral operator on $S^*[D,E]$ and $R[D,E]$

Upon differentiating Bernardi's integral (1.1), we obtain

$$(c+1)f(z) = zF'(z)+cF(z).$$

Logarithmic differentiation now yields

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+c},$$

with $p(z) = zF'(z)/F(z)$.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *hold with $\beta = 1$ and $\gamma = c > -1$. If $f \in S^*[D,E]$, then the function $F(z)$ given by Bernardi's integral* (1.1) *is in $S^*[A,B]$.*

Observe that when $J = 0$, condition (2.1) reduces to the equivalent form

$$(LI - GH)(LH - GI) \geqslant 0. \tag{3.1}$$

Remark 3.1. If $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \leqslant 1)$, then $G = \alpha(1 - c)$, $H = 2\delta(1 + c)$, $I = 2\alpha\delta(1 + k - c)$, $J = 0$ and $L = 1 + c + k$. Since $J = 0$, we need to verify condition (3.1). In this case, $LI - GH = 2\alpha\delta k(2 + k) > 0$. In addition, $LH - GI \geqslant 0$ becomes $(1 + c)(1 + c + k) \geqslant \alpha^2(1 - c)(1 - c + k)$. Clearly this condition holds when $c \geqslant 0$. In the case $-1 < c < 0$, since

$$\frac{(1+c)(2+c)}{(1-c)(2-c)} \leqslant \frac{(1+c)(1+c+k)}{(1-c)(1-c+k)},$$

condition (3.1) holds provided $\alpha^2 \leqslant (1 + c)(2 + c)/((1 - c)(2 - c))$. Thus Theorem 3.1 not only reduces to Theorem 1.1 for $c \geqslant 0$, but also extends it for the case $-1 < c < 0$.

Corollary 3.1. *Let $-1 < c < 0$, $0 < \alpha \leqslant \sqrt{(1 + c)(2 + c)/((1 - c)(2 - c))}$, and δ be as in Theorem* 1.1. *If $f \in S^*[\delta]$, then the function $F(z)$ given by Bernardi's integral* (1.1) *belongs to $S^*[\alpha]$.*

Remark 3.2. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ $(0 \leqslant \alpha, \delta < 1)$, we have $G = 1 - \alpha$, $H = (1 - \delta)(1 + c)$, $I = (1 - \alpha)(1 - \delta)$, $J = 0$ and $L = 1 + c + k$. Since $J = 0$, condition (3.1) reduces to

$$(1 + c)(1 + c + k) - (1 - \alpha)^2 \geqslant 0. \tag{3.2}$$

Since $(1 + c)(1 + c + k) - (1 - \alpha)^2 \geqslant (1 + c)(2 + c) - (1 - \alpha)^2$, inequality (3.2) holds provided $\alpha \geqslant 1 - \sqrt{(1 + c)(2 + c)}$. This condition holds for $c \geqslant (\sqrt{4(\alpha - 1)^2 + 1} - 3)/2$. This yields the following result for the class $S^*(\delta)$.

Corollary 3.2. *Let $\delta := \alpha - (1 - \alpha)/(2 + c - \alpha)$, $f(z) \in S^*(\delta)$ and $F(z)$ be given by Bernardi's integral* (1.1). *If $\alpha_0 \leqslant \alpha < 1$, then $F(z) \in S^*(\alpha)$ for all $c > -1$. Here* $\alpha_0 := (3 + c - \sqrt{(3 + c)^2 - 4})/2$.

Theorem 3.2. *Under the conditions stated in Theorem* 2.1 *with $\beta = 0$ and $\gamma = c + 1$, if $f \in R[D, E]$, then the function $F(z)$ given by Bernardi's integral* (1.1) *is in $R[A, B]$.*

Proof. Since

$$(c + 1)f(z) = zF'(z) + cF(z),$$

we obtain

$$f'(z) = \frac{zF''(z)}{c + 1} + F'(z). \tag{3.3}$$

The result now follows from Theorem 2.1 with $p(z) = F'(z)$, $\beta = 0$ and $\gamma = c + 1$. □

Remark 3.3. For $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \leqslant 1)$, then $G = -\alpha(1 + c)$, $H = 2\delta(1 + c)$, $I = 2\alpha\delta(k - 1 - c)$, $J = 0$ and $L = 1 + c + k$. Condition (3.1) becomes

$$4\alpha\delta^2k^2(1 + c)\left[(1 + c)\left(1 - \alpha^2\right) + k\left(1 + \alpha^2\right)\right] \geqslant 0,$$

which holds for any $c > -1$. This shows that Theorem 3.2 reduces to Theorem 1.2 and that the assertion even holds in the case $-1 < c < 0$.

Remark 3.4. For $A = \delta$, $B = 0$, $D = \alpha$ and $E = 0$ $(0 < \alpha, \delta \leqslant 1)$, we have $G = I = J = 0$, $H = \alpha(1 + c)$, and $L = 1 + c + k$. In this case condition (3.1) holds for any $c > -1$. Thus Theorem 3.2 extends the earlier result of Anbudurai [1, Theorem 2.1, p. 20] even in the case $-1 < c < 0$.

Remark 3.5. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ $(0 \leqslant \alpha, \delta < 1)$, then $G = 0$, $H = (1 - \delta)(1 + c)$, $I = 0$, $J = 0$ and $L = 1 + c + k$. Theorem 3.2 yields the following:

Corollary 3.3. *Let $c > -1$, $1/(2 + c) \leqslant \alpha < 1$ and $\delta := \alpha - (1 - \alpha)/(1 + c)$. If $f(z) \in R_\delta$, then $F(z) \in R_\alpha$.*

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